On reducing the cut ratio to the multicut problem

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Abstract

We compare two multicommodity flow problems, the maximum sum of flow, and the maximum concurrent flow. We show that, for a given graph and a given set of k commodities with specified demands, if the minimum capacity of a multicut is approximated by the maximum sum of flow within a factor of α, for any subset of commodities, then the minimum cut ratio is approximated by the maximum concurrent flow within a factor of $O(\alpha \ln k)$.

1 Introduction

Throughout this note, we are given an undirected graph $G = (V, E)$ where each edge e is assigned a nonnegative real number $c(e)$, called capacity of e . A *flow* from a vertex s called *source* to a vertex t called *sink* is a function α from the set of paths $\mathcal{P}(s,t)$ from s to t into the set of nonnegative real numbers. The *value* of the flow α is $\sum_{P \in \mathcal{P}(s,t)} \alpha(P)$. In a *multicommodity flow* problem, we are given a set Q of k *commodities*. A commodity i is specified by a pair of source and sink (s_i, t_i). A flow in this case is a function α that assigns a nonnegative number to each path $P \in \cup_{i \in Q} \mathcal{P}(s_i, t_i)$. A flow is feasible if the amount of flow traversing any given edge is bounded above by the capacity of that edge, that is, $\sum_{P \ni e} \alpha(P) \leq c(e)$. The value of the flow from s_i to t_i is defined as $\sum_{P \in \mathcal{P}(s_i,t_i)} \alpha(P)$.

A *cut* U separating s and t is a subset of vertices containing s but not containing t. The capacity $\textsc{cap}(U)$ of a cut U is the sum of the capacities of the edges with exactly one endpoint in U. It is intuitively clear that the value of any feasible flow from s to t is at most the capacity of any cut separating s and t. A well known result by Ford and Fulkerson [1] states that in fact, the value of the maximum flow between s and t is equal to the minimum capacity of a cut between s and t. A *multicut* M is a set of edges such that in the graph $(V, E - M)$, each pair of terminals (s_i, t_i) is disconnected.

Two related optimization problems can be defined in multicommodity flow. The first problem consists of maximizing the sum of flows between all pairs of terminals, subject to the capacity constraints. The maximum value for this problem will be called max-flow. The max-flow is obviously upper bounded by the capacity of any multicut. Unlike the case with only one commodity, the max-flow is not equal to the minimum capacity of a multicut. However, an approximate min-max theorem holds in this case: the max-flow is lower bounded by the minimum capacity of a multicut [2] divided by a factor proportional to $\ln k$.

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In the second problem, a *demand* $d(i)$ is associated to each commodity i. We want to calculate the maximum fraction f such that there exists a feasible flow with value $fd(i)$ from s_i to t_i, for each commodity i. The maximum f will be called the maximum concurrent flow. For each cut U, denote by $\textsc{dem}(U)$ the sum of the demands of the commodities with exactly one endpoint in U. The value of f is clearly upper bounded by $\textsc{cap}(U)/\textsc{dem}(U)$, for any cut U. In a series of papers [2, 4, 6, 7], it has been shown that an approximate min-max theorem holds also in this case: the maximum concurrent flow is lower bounded by the minimum cut ratio $\textsc{cap}(U)/\textsc{dem}(U)$ divided by a factor proportional to $\ln^2 k$. The multicommodity flow problem has been used as a basic step to approximately solving a number of NP-hard problems, including graph bisection, deleting the minimum number of edges to make a graph bipartite, and minimum chordalization.

For a subset A of commodities, we denote by $F(A)$ the max-flow of the multicommodity problem where only the commodities in A are considered. Our main result is that it is always possible to reduce the problem of approximating the cut ratio by the maximum concurrent flow to the problem of approximating the max-flow by the minimum multicut. More precisely,

Theorem 1 *Suppose there exists $\beta \in (0,1]$ that may depend on the parameters of the problem such that for any subset A of commodities, the max-flow $F(A)$ is lower bounded by β times the size of the minimum multicut separating all the commodities in A. Then*

$$\Omega(\beta/\ln k) \min_{U\subset V} \frac{\textsc{cap}(U)}{\textsc{dem}(U)} \leq f \leq \min_{U\subset V} \frac{\textsc{cap}(U)}{\textsc{dem}(U)}.$$

Moreover, if there exists a polynomial time algorithm that, for any subset A of commodities, outputs a multicut M separating the commodities in A and such that $\beta\ \textsc{cap}(M) \leq F(A)$, then there exists a polynomial time algorithm that finds a cut U such that

$$\Omega(\beta/\ln k) \frac{\textsc{cap}(U)}{\textsc{dem}(U)} \leq f. \tag{1}$$

For a subset A of commodities, we also denote by $\textsc{dem}(A)$ the sum of demands of the commodities in A. In section 2, we study the relation between the maximum concurrent flow and the minimum max-flow ratio, which is the minimum over all subsets A of commodities of $F(A)/\textsc{dem}(A)$. The maximum concurrent flow is clearly upper bounded by the minimum max-flow ratio. We show that the maximum concurrent flow is lower bounded by the minimum max-flow ratio divided by a factor proportional to $\ln k$.

2 Concurrent flow and max-flow ratio

The commodities will be usually labeled from 1 to k. Define D to be the sum of all the demands.

Theorem 2 *If the demands are integral, we have*

$$\frac{1}{H_D} \min_{A\subseteq Q} \frac{F(A)}{\textsc{dem}(A)} \leq f \leq \min_{A\subseteq Q} \frac{F(A)}{\textsc{dem}(A)}. \tag{2}$$

Proof The upper bound is straightforward. To prove the lower bound, we observe that the dual linear program of the concurrent multicommodity flow problem is

$$I: \quad \text{minimize} \sum_{e\in E} l(e)c(e)$$

subject to:

$$\sum_{i\in Q} d(i)\mathrm{dist}_l(s_i,t_i) \geq 1, \qquad (3)$$
$$l(e) \geq 0, \quad \forall e \in E \qquad (4)$$

where l ranges over the set of length functions on the edges, and $\mathrm{dist}_l(s_i,t_i)$ is the distance between s_i and t_i corresponding to l.

Let l^* be a length function that optimizes this linear program. Relabel the demands by decreasing order of the distance function, so that

$$\mathrm{dist}_{l^*}(s_1,t_1) \geq \mathrm{dist}_{l^*}(s_2,t_2) \geq \cdots \geq \mathrm{dist}_{l^*}(s_k,t_k). \qquad (5)$$

We claim that there exists an index $j \in Q$ such that

$$\mathrm{dist}_{l^*}(s_j,t_j) \geq \frac{1}{H_D \alpha(j)}, \qquad (6)$$

where $\alpha(j) = \sum_{i=1}^{j} d(i)$. Indeed, assume for contradiction that Eq. 6 does not hold for any index $j \in Q$. An easy calculation then shows

$$\begin{aligned}\sum_{j=1}^{k} d(j)\mathrm{dist}_{l^*}(s_j,t_j) &< \frac{1}{H_D}\sum_{j=1}^{k}\frac{d(j)}{\alpha(j)} \\ &= \frac{1}{H_D}\sum_{j=1}^{k}\frac{\alpha(j)-\alpha(j-1)}{\alpha(j)} \\ &\leq \frac{1}{H_D}\sum_{j=1}^{k}\left(\frac{1}{\alpha(j-1)+1}+\frac{1}{\alpha(j-1)+2}+\cdots+\frac{1}{\alpha(j)}\right) \\ &= 1,\end{aligned}$$

since $\alpha(k) = D$. But this contradicts with Eq. 3.

Now, we restrict our attention to the demands $\{(s_i,t_i), i \in Z\}$, where $Z = \{1,2,\ldots j\}$ and j satisfies Eq. 6. We observe that the dual linear program corresponding to the maximum multicommodity flow for this set of demands is

$$\text{minimize} \sum_{e\in E} l(e)c(e)$$

subject to:

$$\mathrm{dist}_l(s_i,t_i) \geq 1, \quad \forall i \in Z, \qquad (7)$$
$$l(e) \geq 0, \quad \forall e \in E \qquad (8)$$

Therefore, for any length function l satisfying Eqs. 7 and 8, we have $F(Z) \leq \sum_{e\in E} l(e)c(e)$ by weak duality. But Eq. 5 shows that the length function $l^*/\mathrm{dist}_{l^*}(s_j,t_j)$ satisfies these equations, and so

$$\mathrm{dist}_{l^*}(s_j,t_j)F(Z) \leq \sum_{e\in E} l^*(e)c(e) = f.$$

Combining this with Eq. 6 and noting that $\alpha(j) = \textsc{dem}(Z)$, we have

$$f \geq \frac{1}{H_D} \frac{F(Z)}{\textsc{dem}(Z)} \geq \frac{1}{H_D} \min_{A \subseteq Q} \frac{F(A)}{\textsc{dem}(A)}.$$

■

Theorem 3 *Without any integrality assumptions on the demands, we have*

$$\frac{1}{1 + \ln(D/d_{\min})} \min_{A \subseteq Q} \frac{F(A)}{\textsc{dem}(A)} \leq f \leq \min_{A \subseteq Q} \frac{F(A)}{\textsc{dem}(A)}. \tag{9}$$

Proof The proof is almost identical to the proof of theorem 2. The only difference is that we now upper bound $(\alpha(j) - \alpha(j-1))/\alpha(j)$ by $\ln \alpha(j) - \ln \alpha(j-1)$, and so

$$\sum_{j=1}^{k} \frac{d(j)}{\alpha(j)} \leq 1 + \sum_{j=2}^{k} (\ln \alpha(j) - \ln \alpha(j-1)) = 1 + \ln(D/d_{\min}).$$

■

We now use the method in [7] to replace the factor $\ln(D/d_{\min})$ by $O(\ln k)$.

Theorem 4 *For any instance of the multicommodity flow problem, we have*

$$\Omega\left(1/\ln k\right) \min_{A \subseteq Q} \frac{F(A)}{\textsc{dem}(A)} \leq f \leq \min_{A \subseteq Q} \frac{F(A)}{\textsc{dem}(A)}. \tag{10}$$

Proof As in [7], we decompose the commodities into groups Q^r, for $r \geq 1$, such that Q^r consists of the commodities with demands between $(4k)^{r-1} d_{\min}$ and $(4k)^r d_{\min}$. Each group gives rise to a different multicommodity flow problem, where only the commodities in group Q^r are considered and the other commodities are ignored. Quantities related to the multicommodity flow problem of the group Q^r will be superscripted by r. The following lemma is central in our analysis:

Lemma 1 *[7] $f \geq (1/4) \min_r f^r$.*

We now apply theorem 3 to each group Q^r. First, we note that the sum of demands of the commodities in each group is upper bounded by a factor of $4k^2$ from the minimum demand in that group. On the other hand, it is clear that $\textsc{dem}^r(A) \leq \textsc{dem}(A)$ for any subset U of Q^r. Therefore, for any group Q^r,

$$f^r \geq \Omega\left(1/\ln k\right) \min_{A \subset Q^r} \frac{F(A)}{\textsc{dem}^r(A)} \geq \Omega\left(1/\ln k\right) \min_{A \subset Q} \frac{F(A)}{\textsc{dem}(A)}.$$

Combining this with lemma 1 achieves the proof. ■

3 Proof of Theorem 1

The upper bound clearly holds. To prove the lower bound, we consider a subset A of demands such that $\Omega(1/\ln k)\, F(A)/\textsc{dem}(A) \leq f$. The existence of A follows from theorem 4. On the other hand, our hypothesis implies the existence of a multicut M separating all demands in A and such that $\beta\, \textsc{cap}(M) \leq F(A)$. We want to find a cut $U \subset V$ such that $\textsc{cap}(U)/\textsc{dem}(U) \leq \textsc{cap}(M)/\textsc{dem}(A)$. Let $U_1, U_2, \ldots$ be the connected components of the graph $(V, E - M)$. First, we observe that

$$\sum_h \textsc{cap}(U_h) \leq 2\textsc{cap}(M). \tag{11}$$

This is because each edge with exactly one of its endpoints in U_h belongs to M, and each edge in M appears at most twice in the sum in the left-hand side. Similarly,

$$\sum_h \textsc{dem}(U_h) \geq 2\textsc{dem}(A). \tag{12}$$

This is because each commodity in A appears two times in the left-hand side, once in the connected component of each of its endpoints. From Eqs. 11 and 12, we see that there exists a component U_h such that $\textsc{cap}(U_h)/\textsc{dem}(U_h) \leq \textsc{cap}(M)/\textsc{dem}(A)$. Therefore,

$$\Omega(\beta/\ln k)\frac{\textsc{cap}(U_h)}{\textsc{dem}(U_h)} \leq \Omega(\beta/\ln k)\frac{\textsc{cap}(M)}{\textsc{dem}(A)} \leq \Omega(1/\ln k)\frac{F(A)}{\textsc{dem}(A)} \leq f.$$

This achieves the proof of the lower bound on f.

The proof of theorems 2, 3, 4 shows that we can find U satisfying Eq. 1 as follows:

1. Separate the commodities into groups Q^r and find r such that f^r is minimized. Set $Q \leftarrow Q^r$.
2. Solve the linear program I, relabel the commodities by decreasing order of their distance function with respect to l^*. Find j such that $\alpha(j)\mathrm{dist}_{l^*}(s_j, t_j)$ is maximized.
3. Find M such that $\beta\, \textsc{cap}(M) \leq \textsc{dem}(Z)$, where $Z = \{1, 2, \ldots, j\}$.
4. Output the connected component U of the graph $(V, E - M)$ that minimizes the ratio $\textsc{cap}(U)/\textsc{dem}(U)$.

■

The linear program I is in fact equivalent to a linear program with $O(E)$ constraints. It can be solved in polynomial time using the interior point method. See [3, 5] for much faster algorithms to approximately solve linear program I. Note that using the algorithm by Garg, Vazirani and Mihalis [2] for finding a multicut within a factor of $O(\ln k)$ from the max-flow, one can deduce immediately from Theorem 1 an algorithm for finding a cut ratio within a factor of $O(\ln^2 k)$ from the maximum concurrent multicommodity flow. Such an algorithm was derived directly in [2, 7], as was mentioned in the introduction.

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