Model-independent lower bound on variance swaps

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Abstract

It is well known that, under a continuity assumption on the price of a stock S, the realized variance on S for maturity T can be replicated by a portfolio of calls and puts maturing at T. This paper assumes call prices on S maturing at T are known for all strikes but makes no continuity assumptions on S. We derive semi-explicit expressions for the supremum lower bound V_{inf} on the hedged payoff, at maturity T, of a long position in the realized variance of S. Equivalently, V_{inf} is the supremum strike K such that an investor with a long position in a variance swap with strike K can ensure to have a non-negative payoff at T. We study examples with constant implied volatilities and with a volatility skew. In our examples, V_{inf} is rather close to the fair variance strike obtained under the continuity assumption.

KEYWORDS: model risk, hedging, sub-replication, realized variance, variance swap

1 Introduction

A variance swap with maturity T and strike K on a stock S is a contract that pays the realized variance minus K at time T. If the value of such a contract is null at inception, we say that K is *the fair variance strike* of the swap. Variance swaps can be used to speculate on future volatility levels and to hedge the volatility exposure of other positions. As noted by Demeterfi et al. (1999), whereas stock options provide exposure to both the direction of the stock price and its volatility, variance swaps provide pure exposure to the volatility level. Carr and Lee (2009) give a detailed history of the variance swaps and other volatility derivatives market.

Unless otherwise specified, we assume that call prices on S for maturity T are known for all strikes. Under a continuity assumption on the stock price, the realized variance can be replicated (Dupire, 1993, Neuberger, 1994) by hedging

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a position in a *log contract* whose payoff at time T is $2T^{-1}\ln(F_0/S_T)$, where S_T is the price of S at T and F_0 is the forward price of the stock. Thus the fair variance strike is equal to the forward price $V_{\log}$ of the log contract. On the other hand, the log contract can itself be statically replicated (Carr and Madan, 1998, Demeterfi et al., 1999, Britten-Jones and A. Neuberger, 2000) by a portfolio of calls and puts maturing at T. Furthermore,

$$V_{\log} = \frac{2}{T}\int_{(0,\infty)} \frac{C(K) - \max(0, F_0 - K)}{K^2}, \tag{1.1}$$

where $C(K)$ is the forward price of a call with maturity T and strike K. Eq. 1.1 has provided the basis for an analytic approximation (Demeterfi et al., 1999) of $V_{\log}$ in the presence of a volatility skew and for the calculation of the VIX index by the Chicago Board Options Exchange (CBOE).

Consider an investor long in the floating leg of a variance swap. The investor is allowed to buy or sell at time 0 call options maturing at T and to take dynamic positions in S during the period $[0, T]$. The hedged payoff of the investor at T is defined as the combined result of his positions. This paper answers the following question: what is the supremum lower bound $V_{\inf}$ on the hedged payoff, at T, of a long position in the realized variance? Equivalently, $V_{\inf}$ is the supremum strike K such that an investor with a long position in a variance swap with strike K can ensure to have a non-negative payoff at T independently of the stock behavior. The answer to the above question is $V_{\log}$ under the continuity assumption since the price of a variance swap with strike $V_{\log}$ is null in this case. This paper, however, makes no assumptions on the stock price process other than being strictly positive. Several authors (Demeterfi et al., 1999, Broadie and Jain, 2008, Carr and Wu, 2009, Carr et al., 2009) have shown that, in the presence of jumps, the price of a variance swap with strike $V_{\log}$ depends on the jumps size, their frequency, and can be significantly positive or negative.

In related work, model-independent bounds on lookback and barrier options have been derived (Hobson, 1998, Brown et al., 2001) in terms of a continuum set of calls with the same maturity. Bounds on prices of basket options (Hobson et al., 2005a,b, Laurence and Wang, 2005, d'Aspremont and El Ghaoui, 2006) and spread options (Laurence and Wang, 2008, 2009) have been established in terms of prices of vanilla options with the same maturity. Kahale (2010) has shown that the calculation of extremum prices of a claim in multiperiod markets consistently with other options prices is reducible to finding an appropriate model of bounded size. Hobson and Neuberger (2010) have recently derived an optimal upper bound for forward start options in terms of vanilla options with corresponding maturities.

The rest of the paper is organized as follows. Section 2 states the remaining definitions and assumptions. Section 3 shows how to obtain lower bounds on variance swaps from a new class of functions, the *V-convex functions*. By building upon the results of section 3, section 4 gives a semi-explicit lower bound on variance swaps via jump processes. Section 5 shows this lower bound is optimal whenever $V_{\inf}$ is finite by constructing martingale measures that asymptotically attain it. It also gives a necessary and sufficient condition for the finiteness of $V_{\inf}$ and, under a regularity condition on the call function, a simplified expression for $V_{\inf}$. Section 6 illustrates our results on two examples. In 6.1, numerical calculations and a simple approximation for $V_{\inf}$ are given if the implied volatilities for maturity T are the same for all strikes. An example with a discrete set of strikes and a volatility skew is treated in 6.2. Section 7 contains concluding remarks. Proofs of some of the results are included in the appendix.

2 Definitions and assumptions

We assume the short risk-free rate $r(t)$ is deterministic and bounded by κ for $t \in [0,T]$. To simplify the notation, we assume that S pays no dividends. It is easy to show that our results still hold in the case where S pays a continuous, deterministic and bounded dividend rate. The call options on S with maturity T are assumed to be arbitrage free and liquid for any non-negative strike K. Thus $C(K)$ is a convex and decreasing function of K (e.g., Davis and Hobson 2007), with $C(0) = F_0$. We further assume that $C'_+(0) = -1$ and that $C(y)$ goes to 0 as y goes to ∞. A standard result (Follmer and Schied, 2004, Lemma 7.23) then implies the existence of a probability measure μ on $[0,\infty)$ such that $C(K) = \int_{(K,\infty)}(z-K)\,d\mu(z)$ for $K \geq 0$, and so $C'_+(K) = -\mu((K,\infty))$. Let $P(K)$ be the forward price of a put with maturity T and strike K. By the call-put parity relation, $C(K) - P(K) = F_0 - K$. For simplicity of presentation, we exclude the trivial case where μ is the Dirac measure δ_{F_0}. Given a subdivision of $[0,T]$ with time-steps $0 = t_0 < \cdots < t_{n+1} = T$, denote by F_i the forward price of the stock at time t_i for maturity T. The realized variance of S over the time-period $[0,T]$ is defined as

$$V[\mathbf{t}] = \frac{1}{T}\sum_{i=0}^{n} \ln^2(\frac{S_{i+1}}{S_i}),$$

where S_i is the value of the stock at time t_i. Let $V_{\inf}[\mathbf{t}]$ be the supremum lower bound on the hedged payoff, at maturity T, of a long position in $V[\mathbf{t}]$. Thus, $V_{\inf} = \inf_{\mathbf{t}} V_{\inf}[\mathbf{t}]$.

3 Variance swaps and V-convex functions

We introduce here the class of V-convex functions and use them to derive lower bounds on $V_{\inf}$. The definition and many of the properties of V-convex functions in this section and in section 4 are inspired from those of convex functions.

Definition 3.1. *A real-valued function g defined on a sub-interval J of $(0,\infty)$ is V-convex if, for $x, y, z \in J$ with $x < z < y$,*

$$\frac{g(x) + \ln^2(x/z) - g(z)}{x - z} \leq \frac{g(y) + \ln^2(y/z) - g(z)}{y - z}. \tag{3.1}$$

Lemma 3.2. *Let g be a real-valued function on an open sub-interval J of $(0,\infty)$. The following conditions are equivalent:*

1. *g is V-convex.*
2. *g is continuous, has a right derivative g'_+ and, for $x, y \in J$ with $x < y$,*

$$g'_+(x)(y - x) \leq g(y) + \ln^2(y/x) - g(x). \tag{3.2}$$

3. For $x > 0$ and $y \in J$,

$$\delta_{x,y;g} = \inf_{v \in J, v > y} \frac{g(v) + \ln^2(v/x) - g(y)}{v - y}$$

is finite and, for $z \in J$,

$$g(y) - g(z) \le \ln^2(z/x) + \delta_{x,y;g}(y - z). \tag{3.3}$$

Proof. See the appendix. □

Note that if we omit the $\ln^2$ terms in Definition 3.1 and Lemma 3.2, we obtain classical properties of convex functions.

Remark 3.3. *The proof of Lemma 3.2 shows that Condition 2 can be replaced by the following one: g is continuous and, for $x \in J$ except for a finite number of points, $g'_+(x)$ exists and Eq. 3.2 holds for $y \in J$ with $x < y$.*

The following variant of results proven in (Breeden and Litzenberger, 1978, Carr and Madan, 1998, Demeterfi et al., 1999) shows how to approximately super-replicate a European derivative meeting certain conditions with standard call options. The idea behind the proof is to super-replicate the derivative payoff with a piece-wise affine function and to observe that such a function is, up to a constant, equal to the payoff of a portfolios of calls.

Lemma 3.4 (Breeden Litzenberger)**.** *Consider a Lipschitz function f on $[0,\infty)$. Then $\int_{(0,\infty)} |f|\, d\mu < \infty$ and, for any $\epsilon > 0$, a European option that pays $f(S_T)$ at maturity T can be super-replicated by a portfolio of a zero-coupon bond and of call options maturing at T with forward price at most $\epsilon + \int_{(0,\infty)} f\, d\mu$.*

Proof. Let $k > 0$ be such that f is k-Lipschitz. Since $|f(z)| \le |f(0)| + kz$, the integral $\int_{(0,\infty)} |f|\, d\mu$ is finite. Consider a strike $K > 0$ such that $C(K) < \epsilon/k$ and let $m = \lceil kK/\epsilon \rceil$. Define f_m as the unique function on $[0,\infty)$ that coincides with f at the points jK/m, $0 \le j \le m$, is affine on the intervals $[jK/m, (j+1)K/m]$, $0 \le j \le m-1$, and on $[K,\infty)$, with slope k on the latter interval. Then

$$f_m(z) - \epsilon - 2k \max(0, z - K) \le f(z) \le f_m(z) + \epsilon \tag{3.4}$$

for $z \ge 0$. Since $f_m + \epsilon$ is piece-wise affine, it can be statically replicated by a portfolio P of a zero-coupon bond and of $m+1$ call options (e.g., Demeterfi et al. 1999) with maturity T. Using Eq. 3.4, we conclude that the forward price $\epsilon + \int_{(0,\infty)} f_m\, d\mu$ of P is at most $4\epsilon + \int_{(0,\infty)} f\, d\mu$. Replacing ϵ with $\epsilon/4$ finishes the proof. □

Theorem 3.5. *Let g be a Lipschitz function on $[0,\infty)$ which is V-convex on $(0,\infty)$ and such that $g(F_0) = 0$. Then $\int_{(0,\infty)} |g|\, d\mu < \infty$ and $V_{\inf} \ge -T^{-1} \int_{(0,\infty)} g\, d\mu$.*

Proof. Consider a subdivision of $[0,T]$ with time-steps $0 = t_0 < \cdots < t_{n+1} = T$. For $0 \le i \le n$, let $Z_i = \exp - \int_{t_{i+1}}^{T} r(u)\, du$. By Condition 3, $\xi_i = \delta_{S_i/Z_i, F_i; g}$ is finite. By Eq. 3.3 and the relation $Z_i F_{i+1} = S_{i+1}$,

$$g(F_i) - g(F_{i+1}) \le \ln^2(S_{i+1}/S_i) + \xi_i (F_i - F_{i+1}), \tag{3.5}$$

and so

$$0 \le g(S_T) + T\ V[\mathbf{t}] + \sum_{i=0}^{n} \xi_i (F_i - F_{i+1}). \tag{3.6}$$

Furthermore, the value of ξ_i is known at time t_i. Fix $\epsilon > 0$. By Lemma 3.4, there is a portfolio P of a zero-coupon bond and of call options maturing at T with forward price at most $\epsilon + \int_{(0,\infty)} g\, d\mu$ that super-replicates g. Consider a long position in the realized variance $V[\mathbf{t}]$. We hedge the position by taking a long position in $T^{-1}P$ at time 0 together with a short position of $T^{-1}\xi_i$ stocks at time t_i and unwinding the latter position at time t_{i+1}. By Eq. 3.6, the payoff at time T of the position in $V[\mathbf{t}]$ combined with the hedges is at least $-T^{-1}(\epsilon + \int_{(0,\infty)} g\, d\mu)$. Thus $V_{\inf} \ge -T^{-1} \int_{(0,\infty)} g\, d\mu$. □

Example 3.6. *Given $a \in (0, F_0]$, the function $g_a(x) = -\ln^2(x/a) 1_{x \ge a}$ is Lipschitz on $[0,\infty)$ and, by Condition 2 of Lemma 3.2, is V-convex on $(0,\infty)$. We conclude that $V_{\inf} \ge T^{-1}(\int_{(a,\infty)} \ln^2(x/a)\, d\mu(x) - \ln^2(F_0/a))$.*

Remark 3.7. *It is easy to show that $\xi_i = g'_+(S_i)$ if interest rates are null.*

4 A lower bound via jump processes

Throughout the rest of the paper, let $a = F_0$, $b = \max\{x \ge 0 : P(x) = 0\}$ and $I = \{y \ge a : C'_-(y) < 0\}$. For $0 < x \le y$ and any function g with a right derivative at x, define

$$\hat{g}(x,y) = g(x) + g'_+(x)(y-x) - \ln^2(y/x).$$

This section establishes a lower bound on $V_{\inf}$ which will be shown to be optimal in Section 5 in the case where $V_{\inf}$ is finite. The main idea behind our approach is to obtain the best lower bound on $V_{\inf}$ achievable through Theorem 3.5. This is a convex optimization problem since the set of V-convex functions is convex.

Assume first that g is known on $[0,a]$. We want to find the smallest V-convex extension (if it exists) of g to (a,∞). Since $\hat{g}(x,y) \le g(y)$ for $0 < x < a < y$ by Eq. 3.2, a natural such extension for g is given by Eq. 4.1 below. Lemma 4.1 shows that, under certain assumptions on g, the proposed extension of g to (a,∞) satisfies the conditions of Theorem 3.5.

Lemma 4.1. *Let g be a V-convex and Lipschitz function on $[0,a]$ which is affine on $[0,b']$, for some $b' \in (0,a)$, with $g(a) = 0$. Define*

$$g(y) = \sup_{x \in (0,a)} \hat{g}(x,y) \tag{4.1}$$

for $y > a$. Then g is Lipschitz and V-convex on $[0,\infty)$.

Proof. Since g is V-convex and Lipschitz on $[0,a]$, it follows from Eq. 3.2 that Eq. 4.1 holds for $y = a$. Thus, for $y \ge a$,

$$g(y) = \sup_{x \in (b'/2,a)} \hat{g}(x,y).$$

This implies that the restriction of g to $[a,\infty)$ is Lipschitz since it is the supremum of functions with uniformly bounded derivatives. Thus g is Lipschitz on $[0,\infty)$. We show that g is V-convex on $(0,\infty)$ by applying Remark 3.3. Since g is V-convex on $[0,a]$, Eq. 3.2 holds for $0 < x < y \leq a$. By Eq. 4.1, it follows that Eq. 3.2 holds for $0 < x < a$ and $x < y$. On the other hand, Example 3.6 shows that the function $y \mapsto -\ln^2(y/x)$ is V-convex on $[a,\infty)$ if $0 < x \leq a$. Since the set of V-convex functions on I is invariant by adding an affine function and by taking the supremum, it follows from Eq. 4.1 that g is V-convex on (a,∞). Thus Eq. 3.2 holds for $a < x < y$. We have thus shown that Eq. 3.2 holds for $0 < x < y$ with $x \neq a$. By Remark 3.3, it follows that g is V-convex on $[0,\infty)$. □

Consider now a decreasing function ϕ from $(b,a]$ to $[a,\infty)$ such that I is the interval spanned by $\phi((b,a])$. Let

$$g(x) = 2\int_{(x,a)} (x-u)\frac{\ln(\phi(u)/u)}{u(\phi(u)-u)}\,du \tag{4.2}$$

for $b \leq x \leq a$ and

$$g(y) = \sup_{x\in(b,a)} \hat{g}(x,y) \tag{4.3}$$

for $y \in I - \{a\}$.

Lemma 4.2. *The function g is finite and V-convex on $[b,a)\cup I$ and, for $x \in (b,a]$,*

$$g(\phi(x)) = \hat{g}(x,\phi(x)). \tag{4.4}$$

For $y \in I$,

$$-\ln^2(y/a) \leq g(y). \tag{4.5}$$

If $\int_{[b,a)\cup I} |g|\,d\mu < \infty$ then $V_{\inf} \geq -T^{-1}\int_{[b,a)\cup I} g\,d\mu$.

Proof. See the appendix. □

To motivate Eq. 4.2, assume that a function g has a continuous second derivative on (b,a), is V-convex on $[b,a)\cup I$ and satisfies Eq. 4.4. Thus the function $u \mapsto \hat{g}(u,\phi(x))$ attains its maximum at x. Taking the derivative with respect to u, it follows that $g''(x) = 2\ln(\phi(x)/x)/(x(x-\phi(x))$, which can be solved via Eq. 4.2. We now justify Eq. 4.4 by assuming that $b > 0$, interest rates are null, and that a function g satisfies Eq. 4.4 and the conditions in Theorem 3.5. The assumptions made in this paragraph are for intuition and are not be used elsewhere in the paper. Let $x_i = a(b/a)^{i/(n+1)}$. By Remark 3.7, if F_i follows a jump process such that $F_{i+1} = F_i$ or $F_i = x_i$ and $F_{i+1} \in \{x_{i+1}, \phi(x_i)\}$, the difference between the right-hand side and the left-hand side of Eq. 3.5 is at most $\ln^2(x_{i+1}/x_i)$. Thus the right-hand side of Eq. 3.6 is bounded by a constant divided by n and $V_{\inf}$ equals $-T^{-1}\int_{[b,a)\cup I} g\,d\mu$. Thus, g would provide the best lower bound on $V_{\inf}$.

The process described above is not, in general consistent with the forward call function C. Under martingale measures consistent with C that will be described in section 5, the forward price follows a process similar to the above-described one for an appropriate ϕ. The following lemma motivates the choice of ϕ.

Lemma 4.3. *Given $i \in [0, n]$ and $0 < x < y$, assume the forward price is a Q-martingale and*

$$\begin{cases} F_i = x \text{ and } S_T \notin (x, y) & \text{or} \\ F_i \in (x, y] \text{ and } F_j = F_i \text{ for } j \geq i, \end{cases} \tag{4.6}$$

Q almost surely. Assume Q is consistent with C and let $C_i(K) = E_Q(\max(F_i - K, 0))$. Then

$$C_i(K) = C(K) - C(y) + \frac{P(x) - C(y)}{y - x}(K - y) \tag{4.7}$$

and

$$C'_-(y) \leq \frac{C(y) - P(x)}{y - x} \leq C'_+(y). \tag{4.8}$$

Furthermore, for any Borel subset A of $[0, \infty)$,

$$Q(F_{i+1} \in A \wedge F_i = x) = Q(F_{i+1} \in A) - Q(F_i \in A) + Q(F_i = x)\delta_x(A). \tag{4.9}$$

Proof. Since F_i and S_T have the same distribution on (x, y) and $C_i(y) = 0$, for $K \in (x, y)$,

$$C_i(K) = C(K) - C(y) + \alpha(K - y),$$

where α is a constant. Since $C_i(x) = F_0 - x$ we conclude, using the call-put parity relation, that

$$\alpha = \frac{P(x) - C(y)}{y - x},$$

which yields Eq. 4.7. Denote by C'_{i-} (resp. C'_{i+}) the left (resp. right) derivative of C_i. The lower bound in Eq. 4.8 follows by observing that $C'_{i-}(y) = C'_-(y) + \alpha$. On the other hand, the relation $Q(F_i > x) \leq Q(S_T \in (x, y])$ implies that $C'_{i+}(x) \geq C'_+(x) - C'_+(y)$. Since $C'_{i+}(x) = C'_+(x) + \alpha$, this yields the upper bound in Eq. 4.8. Finally, Eq. 4.9 follows by observing that

$$\begin{aligned} Q(F_{i+1} \in A \wedge F_i = x) &= Q(F_{i+1} \in A) - Q(F_{i+1} \in A \wedge F_i \neq x) \\ &= Q(F_{i+1} \in A) - Q(F_i \in A \wedge F_i \neq x). \end{aligned}$$

□

Note that Eq. 4.8 implies that the line $(x, P(x))$ and $(y, C(y))$ is tangent at y to the call functions curve. For $(x, y) \in [0, \infty)^2$, define $H(x, y) = C(y) + (x - y)C'_+(y) - P(x)$. Since C is convex, the function $y \mapsto H(x, y)$ is decreasing and right-continuous on $[x, \infty)$. Lemma 4.3 motivates the following choice of ϕ, which will be used throughout the rest of the paper:

$$\phi(x) = \min\{y \geq a : H(x, y) \leq 0\} \tag{4.10}$$

for $x \in (b, a]$. It is easy to see that Eq. 4.8 holds if $y = \phi(x)$. The function ϕ is decreasing on $(b, a]$ and, by the call-put parity relation, $\phi(a) = a$. Finally, set

$$G(x) = \sup_{y > a} \frac{P(x) - C(y)}{y - x} \tag{4.11}$$

for $x \in [0, a]$.

Lemma 4.4. *The interval I is the interval spanned by $\phi((b,a])$. The function G is increasing and continuous on $[0,a]$. For $x \in (b,a)$,*

$$G(x) = \frac{P(x) - C(\phi(x))}{\phi(x) - x}. \tag{4.12}$$

Furthermore, for any nonnegative Borel-measurable function γ on (a,∞),

$$\int_{(b,a)} (\gamma \circ \phi)\, dG = \int_{(a,\infty)} \gamma\, d\mu, \tag{4.13}$$

where the integral in left-hand side is the Stieltjes integral with respect to G. For $x \in (b,a]$,

$$-C'_+(\phi(x)) \le G(x) \le -C'_-(\phi(x)). \tag{4.14}$$

Proof. See the appendix. □

Note that, by Eq. 4.14, if C and ϕ are sufficiently smooth, Eq. 4.13 reduces to the change of variables theorem. Using Eq. 4.10, we conclude the section by an alternative expression for the lower bound in Lemma 4.2.

Lemma 4.5. *The integral $\int_{[b,a)\cup I} |g|\, d\mu$ is finite and $V_{\inf}$ is lower-bounded by*

$$T^{-1} \int_{(b,a)} \ln^2(\phi(x)/x)\, dG(x) = -T^{-1} \int_{[b,a)\cup I} g\, d\mu.$$

Proof. See the appendix. □

5 Martingales attaining the lower bound

Consider the subdivision of $[0,T]$ with time-steps $t_i = iT/(n+1)$, $0 \le i \le n+1$, and let $\Omega = \{F_0\} \times (0,\infty)^{n+1}$ be the set of possible values for the sequence $(F_0, F_1, \dots, F_{n+1})$. We construct martingales consistent with C and that attain, if $V_{\inf}$ is finite, the lower bound in Lemma 4.5 as n goes to infinity. Consider a strictly decreasing sequence x_i, $0 \le i \le n+1$, with $x_0 = a$ and $x_{n+1} = b$. We want F_i to be a Q-martingale such that, Q almost surely, for $0 \le i \le n$, $F_{i+1} = F_i$ or $F_i = x_i$ and $F_{i+1} \in [x_{i+1}, x_i] \cup [\phi(x_i), \phi(x_{i+1})]$. Thus, condition 4.6 holds for $x = x_i$ and $y = \phi(x_i)$. We construct Q by reverse-engineering Eq. 4.7. For $0 \le i \le n$, define

$$C_i(K) = \begin{cases} F_0 - K & \text{if } 0 \le K < x_i \\ C(K) - C(\phi(x_i)) + G(x_i)(K - \phi(x_i)) & \text{if } x_i \le K < \phi(x_i) \\ 0 & \text{if } \phi(x_i) \le K, \end{cases} \tag{5.1}$$

and $C_{n+1} = C$. Using Eq. 4.14, it can be shown that C_i is convex and decreasing on $[0,\infty)$, with $C'_{i+}(0) = -1$. For $0 \le i \le n+1$, let μ_i be the probability measure induced by C_i. In other words, μ_i is the unique probability on $[0,\infty)$ such that $C_i(K) = \int_{(K,\infty)} (z-K)\, d\mu_i(z)$ for $K \ge 0$. Motivated by Eq. 4.9, define the signed measure

$$\nu_i = \delta_{x_i} + \frac{1}{\mu_i(\{x_i\})}(\mu_{i+1} - \mu_i) \tag{5.2}$$

for $0 \leq i \leq n$. Using Eq. 4.14 once again, it can be easily verified that $\mu_i(\{x_i\}) > 0$ and $\nu_i(A) \geq 0$ for any Borel subset A of $[0, \infty)$, and so ν_i is a probability measure on $[0, \infty)$. Let Q be the unique probability on Ω such that (F_i), $0 \leq i \leq n+1$, is a Q-Markov chain with initial probability δ_{F_0} and transition probabilities π_i, $0 \leq i \leq n$, defined as follows:

$$\pi_i(z) = \begin{cases} \nu_i & \text{if } z = x_i \\ \delta_z & \text{otherwise.} \end{cases} \tag{5.3}$$

Lemma 5.1. *For $0 \leq i \leq n+1$ and any Borel subset A of $[0, \infty)$,*

$$Q(F_i \in A) = \mu_i(A). \tag{5.4}$$

Furthermore, (F_i), $0 \leq i \leq n+1$, is a Q-martingale.

Proof. We show Eq. 5.4 by induction on i. The base case clearly holds. If Eq. 5.4 is true for i then $Q(F_i = x_i) = \mu_i(\{x_i\})$ and, for any Borel subset A of $[0, \infty)$,

$$\begin{aligned} Q(F_{i+1} \in A) &= Q(F_i = x_i)\nu_i(A) + Q(F_i \neq x_i \wedge F_{i+1} \in A) \\ &= \mu_i(\{x_i\})\nu_i(A) + Q(F_i \neq x_i \wedge F_i \in A) \\ &= \mu_{i+1}(A) - \mu_i(A - \{x_i\}) + \mu_i(A - \{x_i\}) \\ &= \mu_{i+1}(A). \end{aligned}$$

Thus Eq. 5.4 holds for $i+1$.

On the other hand, by Eqs. 5.3 and 5.4,

$$\begin{aligned} E_Q((F_{i+1} - F_i)1_{F_i = x_i}) &= E_Q(F_{i+1} - F_i) \\ &= C_{i+1}(0) - C_i(0) \\ &= 0. \end{aligned}$$

This shows that F_i is a Q-martingale. □

Lemma 5.2. *For $0 \leq i \leq n+1$ and any nonnegative Borel-measurable function γ on (a, ∞),*

$$\int_{(x_i, a)} (\gamma \circ \phi)\, dG = \int_{(a,\infty)} \gamma\, d\mu_i. \tag{5.5}$$

Proof. See the appendix. □

Theorem 5.3. *If*

$$\int_{(0,\infty)} \ln^2\left(\frac{x}{S_0}\right) d\mu(x) = \infty \tag{5.6}$$

then $V_{\inf} = \infty$. Otherwise, $V_{\inf}$ is finite and equal to

$$-T^{-1} \int_{[b,a)\cup I} g\, d\mu = T^{-1} \int_{(b,a)} \ln^2\left(\frac{\phi(x)}{x}\right) dG(x). \tag{5.7}$$

Proof. Assume first that Eq. 5.6 holds. For $\epsilon > 0$, let

$$f_\epsilon(z) = \begin{cases} 0 & \text{if } z \in [0, \epsilon) \\ \ln^2(z/S_0) & \text{if } z \in [\epsilon, \infty). \end{cases}$$

By Lemma 3.4, a European option that pays $f(S_{n+1})$ at maturity T can be sub–replicated by a portfolio of a zero-coupon bond and of call options maturing at T with forward price at least $\int_{(0,\infty)} f_\epsilon \, d\mu - \epsilon$. By the Cauchy-Schwartz inequality

$$\ln^2 \frac{S_{n+1}}{S_0} \leq (n+1) \sum_{i=0}^{n} \ln^2 \frac{S_{i+1}}{S_i},$$

we conclude that $T(n+1)V_{\text{inf}}[\mathbf{t}] \geq \int_{(0,\infty)} f_\epsilon \, d\mu - \epsilon$. Letting ϵ go to 0 and using the monotone convergence theorem implies that $V_{\text{inf}} = \infty$.

Assume now that $\int_{(0,\infty)} \ln^2(x/S_0) \, d\mu(x)$ is finite. By lemma 4.5, the two sides of Eq. 5.7 are finite, equal and lower bound V_{inf}. We show this lower bound is optimal using the probability Q. By Eq. 5.3, for $0 \leq i \leq n$,

$$\begin{aligned} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) &= Q(F_i = x_i) E_Q\left(\ln^2 \frac{F_{i+1}}{x_i} | F_i = x_i\right) \\ &= \mu_i(\{x_i\}) \int_{(0,\infty)} \ln^2 \frac{z}{x_i} \, d\nu_i(z). \end{aligned}$$

Since ν_i is null on $(0, x_{i+1}) \cup (x_i, a]$, we conclude using Lemma 5.2 that

$$\begin{aligned} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) &\leq \ln^2 \frac{x_{i+1}}{x_i} + \mu_i(\{x_i\}) \int_{(a,\infty)} \ln^2 \frac{z}{x_i} \, d\nu_i(z) \\ &= \ln^2 \frac{x_{i+1}}{x_i} + \int_{(x_{i+1}, x_i]} \ln^2 \frac{\phi(x)}{x_i} \, dG(x) \end{aligned}$$

for $0 \leq i \leq n-1$ and

$$E_Q\left(\ln^2 \frac{F_{n+1}}{F_n}\right) = \int_{(0,x_n)} \ln^2 \frac{z}{x_n} \, d\mu(z) + \int_{(b,x_n]} \ln^2 \frac{\phi(x)}{x_n} \, dG(x).$$

We now set $x_i = a(1/n + (1-1/n)b/a)^{i/n}$ for $0 \leq i \leq n$. It follows that

$$\sum_{i=0}^{n} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) \leq \frac{\alpha}{n} + \int_{(0,x_n)} \ln^2 \frac{z}{x_n} \, d\mu(z) + \int_{(b,a)} \ln^2 \frac{\phi(x)}{x} \, dG(x),$$

where α is a constant independent of n. Since $\int_{(0,\infty)} \ln^2(x/S_0) \, d\mu(x)$ is finite, so is $\int_{(0,\infty)} \ln^2(x/a) \, d\mu(x)$. By Lebesgue's dominated convergence theorem we conclude that

$$\limsup_{n\to\infty} \sum_{i=0}^{n} E_Q\left(\ln^2 \frac{F_{i+1}}{F_i}\right) \leq \int_{(b,a)} \ln^2 \frac{\phi(x)}{x} \, dG(x). \tag{5.8}$$

Since $\ln(S_{i+1}/S_i) \leq \kappa T/(n+1) + \ln(F_{i+1}/F_i)$, it follows from the Cauchy-Schwarz inequality that

$$\sum_{i=0}^{n} \ln^2 \frac{S_{i+1}}{S_i} \leq \frac{(\kappa T)^2}{n} + \sum_{i=0}^{n} \ln^2 \frac{F_{i+1}}{F_i} + \frac{2\kappa T}{\sqrt{n}} \sqrt{\sum_{i=0}^{n} \ln^2 \frac{F_{i+1}}{F_i}}. \tag{5.9}$$

Combining Eqs. 5.8 and 5.9 and using Jensen's inequality shows that

$$\limsup_{n\to\infty} \sum_{i=0}^{n} E_Q \left(\ln^2 \frac{S_{i+1}}{S_i} \right) \leq \int_{(b,a)} \ln^2 \frac{\phi(x)}{x} \, dG(x).$$

Since Q is risk-neutral, this implies that $V_{\text{inf}} \leq T^{-1} \int_{(b,a)} \ln^2(\phi(x)/x) \, dG(x)$. □

Remark 5.4. *It follows from Eqs. 4.2 and 4.5 that* $g(z) \geq g_{\log}(z)$ *for* $z \in [b,a) \cup I$*, where* $g_{\log}(z) = 2(\ln(z/a) - z/a + 1)$*. Hence, when* V_{inf} *is finite,* $V_{\text{inf}} \leq V_{\log}$ *since* $V_{\log} = -T^{-1} \int_{[b,a)\cup I} g_{\log} \, d\mu$*.*

Remark 5.5. *For* $y \in I$*, let* $\psi(y)$ *be the unique solution to the equation* $C(y) + (x-y)C'_-(y) = P(x)$*. It is easy to show that* $\psi(y) \in (b,a]$*. Furthermore,*

$$\phi(u) \leq y \text{ for } u \in (\psi(y), a] \text{ and } \phi(u) \geq y \text{ for } u \in (b, \psi(y)).$$

By a similar proof to that of Eq. 4.4, we conclude that $g(y) = \hat{g}(\psi(y), y)$*.*

Remark 5.6. *Assume* V_{inf} *is finite. Using the proofs and notation of Theorem 3.5 and of Lemma 4.2, it follows that, for any* $\epsilon > 0$*, there is* $b' > b$ *such that an investor with a long position in the floating leg of a variance swap can realize a payoff at least* $V_{\text{inf}} - \epsilon$ *at* T *by buying or selling at time* 0 *a finite number of calls maturing at* T *and dynamically maintaining a short position of* $T^{-1}\delta_{S_i/Z_i, F_i; g_{b'}}$ *stocks at* t_i*.*

Theorem 5.7. *Assume the function* C *is differentiable on* (a,∞) *and* V_{inf} *is finite. Then,*

$$V_{\text{inf}} = T^{-1} \int_I \ln^2 \frac{y}{\psi(y)} \, d\mu(y). \tag{5.10}$$

Proof. Consider $x \in (b,a)$. Eq. 4.10 shows that $H(x, \phi(x)) \leq 0$ and that $H(x,y) > 0$ for $y \in (a, \phi(x))$. Since C is convex and differentiable on (a,∞), C' is continuous on (a,∞). We conclude that $H(x,\phi(x)) = 0$. Equivalently, $\psi(\phi(x)) = x$ for $x \in (b,a)$. By applying Eq. 4.13 to the function γ defined on (a,∞) with

$$\gamma(y) = \begin{cases} \ln^2(y/\psi(y)) & \text{if } y \in I - \{a\} \\ 0 & \text{otherwise,} \end{cases}$$

it follows that

$$\int_{(b,a)} \ln^2 \frac{\phi(x)}{x} \, dG(x) = \int_{I-\{a\}} \ln^2 \frac{y}{\psi(y)} \, d\mu(y).$$

We conclude the proof by noting that $\psi(a) = a$ and using Theorem 5.3. □

Eq. 5.10 has a simple intuitive interpretation: when n goes to infinity, the forward price follows a process where it may jump from $\psi(y)$ to y.

6 Examples

We study an example with constant implied volatilities and an example with a volatility skew.

σ	10%	15%	20%	25%	30%	35%
$\sqrt{V_{\text{inf}}}$	9.782%	14.510%	19.129%	23.641%	28.044%	32.340%
$\sigma - \frac{c}{2}\sigma^2\sqrt{T}$	9.782%	14.509%	19.128%	23.637%	28.038%	32.329%

Table 1: $\sqrt{V_{\text{inf}}}$ and its approximation obtained via Theorems 5.7 and 6.1 when $T = 0.25$.

6.1 A constant implied volatilities example

This subsection assumes that the implied volatilities for maturity T and all strikes are equal to a constant σ. For $u \geq 0$, denote by $\theta(u)$ the unique solution to the equation $N'(u) - N'(\theta) = \theta(N(\theta) + N(-u))$ in the interval $[-u, 0]$.

Theorem 6.1. *If the call prices for all strikes and maturity T are equal to the Black-Scholes prices with volatility σ, then $V_{\text{inf}} = \sigma^2 - c\sigma^3\sqrt{T} + O(\sigma^4)$ as $\sigma \to 0$, where*

$$c = \frac{1}{2}\int_0^\infty u(u - \theta(u))^2 N'(u)\, du \approx 0.8721.$$

Proof. See the appendix. □

Table 1 shows values of $\sqrt{V_{\text{inf}}}$ calculated numerically via Theorem 5.7 when $T = 0.25$ and compares them with the approximation $\sigma - c\sigma^2\sqrt{T}/2$.

6.2 A discrete set of strikes example

In practice, European call (or, equivalently, put) options prices with a given maturity T are known only for a finite set of positive strikes $K_1 < \cdots < K_l$. Demeterfi et al. (1999) have calculated $V_{\log}$ in this case by approximating the function $g_{\log}$ with a piecewise affine function. This is computationally equivalent to extending the call prices to all strikes via a piecewise affine interpolation. We use a similar approach to calculate V_{inf}. Assume the sequence $C(0), C(K_1), \ldots, C(K_l)$ is strictly convex and strictly decreasing with respect to the strike, where $C(0) = F_0$, and that $C(K_1) \geq F_0 - K_1$. By the necessary and sufficient conditions for the absence of arbitrage among the call prices established by Davis and Hobson (2007), it can be shown there are two strikes $0 < K_0 < K_1$ and $K_{l+1} > K_l$ such that the following extension of C to all strikes is arbitrage-free: $C(K) = F_0 - K$ for $K \leq K_0$, $C(K) = 0$ for $K \geq K_{l+1}$, and

$$C(K) = C(K_i) + \frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i}(K - K_i)$$

for $K_i < K < K_{i+1}$. Furthermore, the support of μ is included in the set $\{K_0, \ldots, K_{l+1}\}$, with

$$\mu(K_0) = 1 - \frac{C(K_1) - C(K_0)}{K_1 - K_0},$$

$$\mu(K_i) = \frac{C(K_{i+1}) - C(K_i)}{K_{i+1} - K_i} - \frac{C(K_i) - C(K_{i-1})}{K_i - K_{i-1}}$$

for $1 \leq i \leq l$, and

$$\mu(K_{l+1}) = \frac{C(K_l)}{K_{l+1} - K_l}.$$

We now can apply Theorem 5.3 to calculate V_{inf}. The function ϕ can be calculated by inspection via Eq. 4.10, which implies that $\text{Im}(\phi) \subseteq \{F_0\} \cup \text{Supp}(\mu)$. On the other hand, Eq. 4.13 shows that, for $y \in \text{Supp}(\mu) \cap (F_0, \infty)$, the set $\phi^{-1}(\{y\})$ is a sub-interval of $(K_0, F_0]$ with a non-empty interior. Thus, an element x of $\phi^{-1}(\{y\})$ can be found using the bisection method. We can then calculate $g(y)$ via Eq. 4.4.

In our numerical example in table 2, we assume that $K_i = 35 + 5i$ for $1 \leq i \leq l$, with $l = 22$. We chose $K_0 = 35$ and $K_{l+1} = 150$. The function g has been calculated on $\text{Supp}(\mu) = \{K_0, \dots, K_{l+1}\}$ via Eqs. 4.2 and 4.3, and G has been evaluated via Eq. 4.12. The integrals in Eqs. 4.2, 4.3 and 5.7 have been calculated using the midpoint method with 100000 points. The difference between the right-hand and left-hand sides of Eq. 5.7 is of order 10^{-7}. Eq. 5.7 shows that $V_{\text{inf}} = (24.263\%)^2$. In comparison, we have calculated V_{log} via the equation $V_{\text{log}} = \sum_{i=0}^{l+1} \mu(K_i) g_{\text{log}}(K_i)$ and found that $V_{\text{log}} = (25.608\%)^2$.

Table 3 compares the values of V_{inf} and of V_{log} using the same implied volatility function $\sigma_{\text{imp}}(K) = 0.45 - 0.002K$ as in table 2 and different values for the strikes spacing. We do the same comparison in table 4 using a constant implied volatility function $\sigma_{\text{imp}}(K) = 0.25$. In the limit case $\Delta K = 0$, V_{inf} has been extracted from table 1.

7 Concluding remarks

Alternative semi-explicit expressions for the optimal lower bound V_{inf} on the hedged payoff of a long position in the realized variance have been given. We have assumed that call prices on S with maturity T are known for all strikes but did not make any continuity assumptions on S.

When V_{inf} is finite, for any $\epsilon > 0$, an explicit hedging scheme allows an investor with a long position in the floating leg of a variance swap to realize a payoff at least $V_{\text{inf}} - \epsilon$ at T. Furthermore, the expected realized variance is at most $V_{\text{inf}} + \epsilon$ under an explicitly constructed martingale.

Numerical values and an approximation for V_{inf} have been given when the implied volatilities for maturity T are constant, and a numerical example with a discrete set of strikes and a volatility skew has been treated. Quite surprisingly, even though the models we have used to attain V_{inf} are jump processes, the values of V_{inf} in our examples are rather close to the fair variance strikes obtained under the continuity assumption.

A Proof of Lemma 3.2

$1 \Rightarrow 2$. Assume that g is V-convex. Since the function $u \mapsto \ln^2 u + 4 \ln u$ is concave on an open interval containing 1,

$$\text{(A.1)} \qquad \frac{\ln^2(y/z) + 4\ln(y/z)}{y - z} \leq \frac{\ln^2(x/z) + 4\ln(x/z)}{x - z}$$

for $0 < (1+\epsilon)^{-1}z < x < z < y < (1+\epsilon)z$, where ϵ is a sufficiently small positive constant. Combining Eqs. 3.1 and A.1 shows that, for any $z \in J$, the function $g(v) - 4\ln v$ is convex on the interval $(z, (1+\epsilon)z) \cap J$. Consequently,

Strike	Implied Volatility	Call price	μ	g	$g_{\log}$
35			0.000000	-0.5770	-0.8062
40	37%	60.199502	0.000002	-0.4725	-0.6386
45	36%	55.224448	0.000015	-0.3832	-0.5025
50	35%	50.249468	0.000088	-0.3066	-0.3913
55	34%	45.274923	0.000394	-0.2411	-0.3002
60	33%	40.302340	0.001411	-0.1855	-0.2257
65	32%	35.336777	0.004169	-0.1387	-0.1651
70	31%	30.391954	0.010436	-0.0998	-0.1164
75	30%	25.499053	0.022568	-0.0681	-0.0779
80	29%	20.718432	0.042685	-0.0431	-0.0483
85	28%	16.150173	0.071128	-0.0242	-0.0266
90	27%	11.935778	0.104687	-0.0109	-0.0117
95	26%	8.242208	0.135842	-0.0030	-0.0031
100	25%	5.224458	0.154436	0.0000	0.0000
105	24%	2.975040	0.152160	-0.0019	-0.0019
110	23%	1.482628	0.127838	-0.0081	-0.0084
115	22%	0.626216	0.089568	-0.0179	-0.0190
120	21%	0.215409	0.050812	-0.0304	-0.0334
125	20%	0.057392	0.022458	-0.0447	-0.0512
130	19%	0.011107	0.007359	-0.0597	-0.0723
135	18%	0.001437	0.001677	-0.0743	-0.0963
140	17%	0.000111	0.000245	-0.0865	-0.1231
145	16%	0.000004	0.000021	-0.0947	-0.1524
150			0.000001	-0.0970	-0.1841

Table 2: The spot price is \$100, implied volatilities are given for European call options with maturity $T = 0.25$ and strikes ranging from \$40 to \$145, spaced \$5 apart. There are no dividends and the continuously compounded risk-free interest rate with maturity T is 2%.

ΔK	0.1	1	5
V_{inf}	$(23.955\%)^2$	$(23.967\%)^2$	$(24.263\%)^2$
$V_{\log}$	$(25.267\%)^2$	$(25.280\%)^2$	$(25.608\%)^2$

Table 3: Variance rates in the presence of skew. Strikes range in the interval $[40, 200]$.

ΔK	0	0.1	1	5
V_{inf}	$(23.641\%)^2$	$(23.641\%)^2$	$(23.653\%)^2$	$(23.951\%)^2$
$V_{\log}$	$(25.000\%)^2$	$(25.000\%)^2$	$(25.014\%)^2$	$(25.344\%)^2$

Table 4: Variance rates with flat volatility. Strikes range in the interval $[40, 200]$.

g is continuous and has a right derivative. Consider now $x, y \in J$, with $x < y$. Eq. 3.2 follows by taking limits as $z \to x^+$ in Eq. 3.1.

$2 \Rightarrow 3$. Assume 2. holds. We first show that, for $x, y, z \in J$ with $x < z < y$,

$$\frac{g(z) - g(x)}{z - x} \leq \frac{g(y) - g(z)}{y - z} + \frac{\ln^2(y/x)}{y - x}. \tag{A.2}$$

Fix $x, y \in J$, with $x < y$. For $z \in [x, y]$, let

$$h(z) = g(z) - g(x) - \frac{g(y) - g(x)}{y - x}(z - x) + \frac{\ln^2(y/x)}{(y - x)^2}(z - y)(z - x).$$

Since the map $u \mapsto (y - u)^{-1}\ln(y/u)$ is decreasing on $(0, y)$,

$$\begin{aligned} h(z) + h'_+(z)(y - z) &= g(z) + g'_+(z)(y - z) - g(y) - \frac{\ln^2(y/x)}{(y - x)^2}(y - z)^2 \\ &\leq \ln^2(y/z) - \frac{\ln^2(y/x)}{(y - x)^2}(y - z)^2 \\ &\leq 0. \end{aligned} \tag{A.3}$$

Let v be the first point in the interval $[x, y]$ where h attains its maximum. If $h(v) > 0$ then h is positive on an interval $[u, v]$, where $u \in (x, v)$. By Eq. A.3, the function h is decreasing on $[u, v]$, leading to a contradiction. We conclude that $h(z) \leq 0$ for $x \leq z \leq y$ which, after some simplifications, is equivalent to Eq. A.2.

Now, for $x > 0$ and $u, y, v \in J$ with $u < y < v$, the Cauchy-Schwarz inequality

$$\left(\sqrt{v - y}\frac{\ln(v/x)}{\sqrt{v - y}} + \sqrt{y - u}\frac{\ln(x/u)}{\sqrt{y - u}}\right)^2 \leq (v - u)\left(\frac{\ln^2(v/x)}{v - y} + \frac{\ln^2(x/u)}{y - u}\right)$$

can be rewritten as

$$\frac{\ln^2(v/u)}{v - u} \leq \frac{\ln^2(v/x)}{v - y} + \frac{\ln^2(u/x)}{y - u}. \tag{A.4}$$

Combining Eq. A.2, applied to (u, y, v), and Eq. A.4 yields

$$\frac{g(y) - g(u) - \ln^2(u/x)}{y - u} \leq \frac{g(v) - g(y) + \ln^2(v/x)}{v - y}.$$

This implies that $\delta_{x,y;g}$ is finite and that Eq. 3.3 holds.

$3 \Rightarrow 1$. Assume 3. holds. Consider $x, z, y \in J$ with $x < z < y$. It follows from Eq. 3.3 that

$$\frac{g(x) + \ln^2(x/z) - g(z)}{x - z} \leq \delta_{z,z;g}.$$

Since

$$\delta_{z,z;g} \leq \frac{g(y) + \ln^2(y/z) - g(z)}{y - z},$$

we conclude that Eq. 3.1 holds. □

B Proof of Lemma 4.2

We first observe that $\phi(a) = a$. Eq. 4.2 shows that g is continuous on $[b, a]$ and that, for $x \in (b, a)$,

$$g'(x) = 2\int_{(x,a)} \frac{\ln(\phi(u)/u)}{u(\phi(u) - u)} du. \tag{B.1}$$

It follows from Eqs. 4.2 and B.1 that, for $b < x < z < a$ and $y \geq z$,

$$\hat{g}(z, y) - \hat{g}(x, y) = 2\int_{(x,z)} \frac{y - u}{u}\left(\frac{\ln(y/u)}{y - u} - \frac{\ln(\phi(u)/u)}{\phi(u) - u}\right) du. \tag{B.2}$$

Since the map $v \mapsto \frac{\ln v}{v-1}$ is decreasing on $(1, \infty)$, Eq. B.2 implies that $\hat{g}(x, y) \leq \hat{g}(y, y)$ for $b < x < y < a$ and so Eq. 3.2 holds for $b < x < y < a$. Thus g is V-convex on $[b, a]$. Since ϕ is decreasing, Eq. B.2 implies that $\hat{g}(x, y) \leq \hat{g}(z, y)$ for $b < x < z < a \leq y \leq \phi(z)$, and so $g(y)$ is finite. Eq. 4.4 follows from Eq. B.2 by a similar argument.

Given $b' \in (b, a)$, let

$$g_{b'}(z) = \begin{cases} g(b') + g'(b')(z - b') & \text{for } z \in [0, b') \\ g(z) & \text{for } z \in [b', a] \\ \sup_{x\in[b',a)} \hat{g}(x, z) & \text{for } z \in (a, \infty). \end{cases}$$

By Eq. B.1, the function $g_{b'}$ is Lipschitz on $[0, a]$. On the other hand, by Condition 2 of Lemma 3.2 and since g is V-convex on $[b, a]$, the function $g_{b'}$ is V-convex on $[0, a]$. By Lemma 4.1, it follows that $g_{b'}$ is Lipschitz and V-convex on $[0, \infty)$ and so, by Theorem 3.5,

$$V_{\text{inf}} \geq -T^{-1} \int_{[b,a)\cup I} g_{b'}\, d\mu. \tag{B.3}$$

On the other hand, by Eq. B.1, g is concave and increasing on (b, a) and so

$$g(x) \leq g_{b'}(x) \leq 0 \tag{B.4}$$

for $b \leq x \leq a$. Also, since $g'(x)$ goes to 0 as x goes to a, $x < a$,

$$-\ln^2(y/a) \leq g_{b'}(y) \leq g(y)$$

for $y \in I$. Thus, the functions $g_{b'}$ are dominated by a μ-integrable function on $[b, a) \cup I$. Furthermore, from Eq. B.4 and the bound $g_{b'}(b) \leq g(b')$, the function $g_{b'}$ converges pointwise to g on $[b, a) \cup I$ as b' goes to b. The lemma thus follows from Eq. B.3 and Lebesgue's dominated convergence theorem. □

C Proof of Lemma 4.4

The function G is increasing and continuous on $[0, a]$ since it is the supremum of increasing and continuous functions on $[0, a]$ with uniformly bounded right derivatives on any closed sub-interval of $[0, a)$. Fix $x \in (b, a)$. Eq. 4.12 follows from Eq. 4.10 by observing that the right derivative of the function $y \mapsto \frac{P(x)-C(y)}{y-x}$ is positive for $a < y < \phi(x)$ and at most 0 for $y > \phi(x)$.

We first show Eq. 4.13 in the case where $\gamma = 1_{(v,\infty)}$, with $v \geq a$. Let

$$u = \min\{x \geq 0 : H(x, v) = 0\}. \tag{C.1}$$

By the convexity of C, the function $y \mapsto C(y)+(u-y)C'_+(v)$ attains its minimum at v, and $u \leq a$. Thus, by Eq. C.1,

$$\min_{y \geq a} C(y) + (u - y)C'_+(v) = P(u),$$

and so

$$G(u) = -C'_+(v). \tag{C.2}$$

On the other hand, it follows from Eqs. 4.10 and C.1 that, for $x \in (b, a)$,

$$\begin{aligned} v < \phi(x) \quad &\Leftrightarrow \quad H(x, v) > 0 \\ &\Leftrightarrow \quad x < u. \end{aligned}$$

Thus, by Eq. C.2 and since $G(b) = 0$, Eq. 4.13 holds when $\gamma = 1_{(v,\infty)}$. Consequently, Eq. 4.13 holds when γ is the indicator function of any Borel subset of (a, ∞). Since any nonnegative Borel-measurable function on (a, ∞) is a pointwise limit of an increasing sequence of non-negative simple functions on (a, ∞), this concludes the proof of Eq. 4.13.

Eq. 4.14 follows by applying Eq. 4.13 in the case where $\gamma = 1_{(\phi(x),\infty)}$ or $\gamma = 1_{[\phi(x),\infty)}$.

We now prove the first assertion in the lemma. Consider $y \in I - \{a\}$. Since the right-hand side of Eq. 4.13 is $-C'_-(y)$ when $\gamma = 1_{[y,\infty)}$, there is $x \in (b, a)$ with $y \leq \phi(x)$. Thus, y belongs to the interval spanned by $\phi((b, a])$. Conversely, consider $x \in (b, a)$ and assume for contradiction that $C'_-(\phi(x)) = 0$. By Eqs. 4.14 and 4.12, we conclude that $P(x) = C(\phi(x)) = 0$, leading to a contradiction. Thus, I contains the interval spanned by $\phi((b, a])$. □

D Proof of Lemma 4.5

By Lemma 4.4, for $u \in (b, a)$,

$$\begin{aligned} \int_{(b,u)} (\phi(x) - \phi(u))\, dG(x) &= \int_{(b,a)} \max(0, \phi(x) - \phi(u))\, dG(x) \\ &= \int_{(a,\infty)} \max(0, y - \phi(u))\, d\mu(y) \\ &= C(\phi(u)). \end{aligned}$$

Since

$$\int_{(b,u)} (\phi(x) - u)\, dG(x) = \int_{(b,u)} (\phi(x) - \phi(u))\, dG(x) + (\phi(u) - u)G(u),$$

we conclude, using Eq. 4.12, that

$$\int_{(b,u)} (\phi(x) - u)\, dG(x) = P(u).$$

Equivalently,

$$\text{(D.1)} \qquad \int_{(b,a)} (p_u + (\phi - \mathcal{I}){p_u}'_+)\, dG = -\int_{[b,a)} p_u\, d\mu,$$

where $p_u(x) = \max(0, u - x)$, ${p_u}'_+$ is the right derivative of p_u with respect to x and $\mathcal{I}$ is the identity function. Since, for $x \in (b, a)$,

$$g(x) = \int_{(b,a)} \omega(u) p_u(x)\, du$$

and

$$g'(x) = \int_{(b,a)} \omega(u) {p_u}'_+(x)\, du,$$

where $\omega(u) = -2\frac{\ln(\phi(u)/u)}{u(\phi(u)-u)}$, it follows from Eq. D.1 and Fubini's theorem that

$$\text{(D.2)} \qquad \int_{(b,a)} (g + (\phi - \mathcal{I})g')\, dG = \int_{[b,a)} -g\, d\mu.$$

Both integrands in Eq. D.2 are positive and, since g is continuous, $\int_{[b,a)} -g\, d\mu$ is finite. Eqs. D.2 and 4.4 show that the integral $\int_{(b,a)} \max(0, g \circ \phi)\, dG$ is finite. Also, by setting $g(y) = 0$ for $y \in (a, \infty) - I$, it follows from Eq. 4.5 that the integral $\int_{(a,\infty)} \max(0, -g)\, d\mu$ is finite. Thus, by Lemma 4.4,

$$\text{(D.3)} \qquad \int_{(b,a)} (g \circ \phi)\, dG = \int_{(a,\infty)} g\, d\mu.$$

Combining Eqs. D.2, 4.4 and D.3 concludes the proof. □

E Proof of Lemma 5.2

The proof is similar to that of Eq. 4.13, which is identical to Eq. 5.5 when $i = n + 1$. Assume now that $0 \le i \le n$. We first observe that both sides of Eq. 5.5 are null in the case where $\gamma = 1_{(v,\infty)}$, with $v \ge \phi(x_i)$. Similarly, when $\gamma = 1_{(a,v]}$, with $v < \phi(x_i)$, then by Eq. 4.13

$$\begin{aligned} \int_{(x_i,a)} (\gamma \circ \phi)\, dG &= \int_{(b,a)} (\gamma \circ \phi)\, dG \\ &= \mu((a, v]) \\ &= \mu_i((a, v]) \\ &= \int_{(a,\infty)} \gamma\, d\mu_i. \end{aligned}$$

Thus Eq. 5.5 also holds in this case. Finally, since $C(a) = P(a)$ and C is convex, it follows from Eq. 4.11 that $G(a) = -C'_+(a)$, and so Eq. 5.5 holds when γ is constant and equal to 1 on (a, ∞). Consequently, Eq. 5.5 holds when γ is the indicator function of any Borel subset of (a, ∞), and so it holds for any nonnegative Borel-measurable function γ on (a, ∞). □

F Proof of Theorem 6.1

Assume for simplicity that $F_0 = T = 1$. Then, for $v \le 0 \le u$ and $\sigma \ge 0$,

$$\text{(F.1)}\quad H(e^{\sigma v}, e^{\sigma u}) = N(-u + \frac{\sigma}{2}) + N(v - \frac{\sigma}{2}) - e^{\sigma v}(N(-u - \frac{\sigma}{2}) + N(v + \frac{\sigma}{2})),$$

where $N(z) = (2\pi)^{-1/2}\int_{-\infty}^{z} \exp(-w^2/2)\,dw$. It follows that $H(1, e^{\sigma u}) \le 0 \le H(e^{-\sigma u}, e^{\sigma u})$ and so

$$\text{(F.2)}\qquad \psi(e^{\sigma u}) \in [e^{-\sigma u}, 1].$$

Let

$$h(v, u) = N'(u) - N'(v) - vN(-u) - vN(v)$$

and

$$\eta(\sigma; v, u) = e^{-\sigma v/2} H(e^{\sigma v}, e^{\sigma u}).$$

Eq. F.1 shows that the definitions of H and η can be extended to any real number σ and the function η is odd with respect to σ. On the other hand, $\partial\eta(\sigma; v, u)/\partial\sigma = h(v, u)$ at $\sigma = 0$. Furthermore, an easy calculation shows that there is a constant $\gamma > 0$ such $|\partial^3\eta(\sigma; v, u)/\partial\sigma^3| \le \gamma e^{2u} N'(v)$ for $-u \le v \le 0 \le u$ and $0 \le \sigma \le 1$ and so, by Taylor-Lagrange theorem,

$$\text{(F.3)}\qquad |\eta(\sigma; v, u) - \sigma h(v, u)| \le \gamma\sigma^3 e^{2u} N'(v)/6.$$

Let us now assume that $H(e^{\sigma v}, e^{\sigma u}) = 0$ and $\gamma\sigma^2 e^{4u} \le 1$, with $v \le 0 \le u$ and $0 \le \sigma \le \min(1, 1/\gamma)$. By Eq. F.2, $-u \le v \le 0$ and so, by Eq. F.3, $|h(v, u)| \le \gamma\sigma^2 e^{2u} N'(v)/6$. We show that

$$\text{(F.4)}\qquad |v - \theta(u)| \le \gamma\sigma^2 e^{3u}.$$

Since the function $N(z) - e^z N'(z)$ has a positive derivative and goes to 0 as z goes to $-\infty$,

$$\text{(F.5)}\qquad e^v N'(v) \le N(v).$$

We now distinguish two cases:

- $v \le \theta(u)$: since $\partial h(v, u)/\partial v = -N(v) - N(-u)$ and h is concave with respect to v,

  $$h(\theta(u), u) - h(v, u) \le -(\theta(u) - v)(N(v) + N(-u)).$$

  Thus $(\theta(u) - v)N(v) \le \gamma\sigma^2 e^{2u} N'(v)/6$. By Eq. F.5, we conclude that Eq. F.4 holds in this case.

- $v > \theta(u)$: Let $z = v - \gamma\sigma^2 e^{3u}$. By Taylor-Lagrange theorem and Eq. F.5,

  $$\begin{aligned} h(z, u) &\ge h(v, u) + (v - z)(N(v) + N(-u)) - \frac{(v - z)^2}{2} N'(v) \\ &\ge N'(v)(-\gamma\sigma^2 e^{2u}/6 + (v - z)e^{-u} - \frac{(v - z)^2}{2}) \\ &\ge 0 \\ &\ge h(v, u). \end{aligned}$$

  Thus Eq. F.4 holds again in this case.

Eq. F.4 implies that, for $0 \le \sigma \le \min(1, 1/\gamma)$ and $0 \le u \le -\ln(\gamma\sigma^2)/4$,

$$|\ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} - \sigma^2(u - \theta(u))^2| \le 4\gamma\sigma^4 u e^{3u}. \tag{F.6}$$

By Theorem 5.7, Eqs. F.2 and F.6,

$$\begin{aligned}
V_{\text{inf}} &= \int_1^\infty \ln^2 \frac{y}{\psi(y)} C''(y)\, dy \\
&= \int_0^\infty \ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} N'(u + \sigma/2)\, du \\
&= \int_0^{-\ln(\gamma\sigma^2)/4} \ln^2 \frac{e^{\sigma u}}{\psi(e^{\sigma u})} N'(u + \sigma/2)\, du + O(\sigma^4) \\
&= \sigma^2 \int_0^\infty (u - \theta(u))^2 N'(u + \sigma/2)\, du + O(\sigma^4) \\
&= \sigma^2 \int_0^\infty (u - \theta(u))^2 N'(u)\, du - c\sigma^3 + O(\sigma^4).
\end{aligned}$$

On the other hand, by the implicit function theorem the function θ has a continuous derivative for $u > 0$. Thus

$$\begin{aligned}
\int_0^\infty (u - \theta(u))^2 N'(u)\, du &= \frac{1}{2} + 2\int_0^\infty \theta(u) N''(u)\, du + \int_0^\infty \theta(u)^2 N'(-u)\, du \\
&= \frac{1}{2} - 2\int_0^\infty \theta'(u)(N'(u) - \theta(u)N(-u))\, du \\
&= \frac{1}{2} - 2\int_0^\infty \theta'(u)(N'(\theta(u)) + \theta(u)N(\theta(u)))\, du \\
&= \frac{1}{2} + 2\int_{-\infty}^0 (N'(z) + zN(z))\, dz \\
&= 1.
\end{aligned}$$

This concludes the proof of the theorem. □

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